

ADEQUACY OF LINK FAMILIES

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Abstract

Using computer calculations and working with representatives of pretzel tangles we established general adequacy criteria for different classes of knots and links. Based on adequate graphs obtained from all Kauffman states of an alternating link we defined a new numerical invariant: adequacy number, and computed adequacy polynomial which is the invariant of alternating link families. Adequacy polynomial distinguishes (up to mutation) all families of alternating knots and links whose generating link has at most $n = 12$ crossings.

Keywords: Adequate diagram, adequate link, semi-adequate link, inadequate link, adequacy number, adequacy polynomial

1. Introduction

First we give a brief overview of the properties of adequate, semi-adequate and inadequate link diagrams and their corresponding links. In this paper, we will consider only prime links.

Let D be a diagram of an unoriented link L framed in a 3-ball B^3 . A Kauffman state of a diagram D is a function from the set of crossings of D to the set of signs $\{+1, -1\}$. Graphical interpretation is smoothing each crossing of D by introducing markers according to the convention illustrated in Fig. 1. A *state diagram* D_s is a system of circles obtained by smoothing all crossings of D [PrAs]. The set of circles in D_s , which are called *state circles*, is denoted by $C(D)$. Points of the state circles corresponding to a smoothed crossing are called *touch-points*. The number of touch-points belonging to a state circle $c \in C(D)$ is called the *length* of c .

Kauffman states s_+ and s_- with all $+$ or all $-$ signs are called *special states*, and their corresponding state diagrams D_{s_+} and D_{s_-} are called *special diagrams*. All other Kauffman states with both $+$ or $-$ signs are called *mixed states*, and to them correspond *mixed state diagrams*.

Definition 1. A diagram D is *s-adequate* if two arcs at every touch-point of D_s belong to different state circles. In particular, a diagram D is *+adequate* or *-adequate* if it is s_+ or s_- adequate, respectively. If a diagram is neither *+adequate* nor *-adequate* it is called *inadequate*. If a diagram is both *+adequate* and *-adequate*, it is called *adequate*, and if it is only *+adequate* or *-adequate*, it is called *semi-adequate* [LiThi,Li].

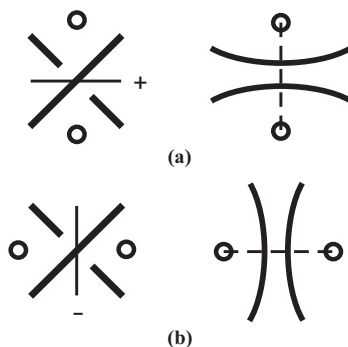


Figure 1: (a) $-$ marker; (b) $+$ marker. The broken lines represent the edges of the associated graph G_s connecting state circles (represented by dots).

To every state diagram D_s we associate the graph G_s , whose vertices are state circles of D_s and edges are lines connecting state circles via smoothed crossings in D . Now we can restate Definition 1 in terms of G_s : D is s -adequate if G_s is loopless. A state graph G_s is called adequate if D_s is s -adequate.

Definition 2. A link is *adequate* if it has an adequate ($+$ adequate and $-$ adequate) diagram. A link is *semi-adequate* if it has a $+$ or $-$ adequate diagram. A link is *inadequate* if it is neither $+$ or $-$ adequate [LiThi,Li].

The mirror image of a diagram transforms the $+$ adequacy into $-$ adequacy and *vice versa*.

Definition 3. A link that has one $+$ adequate diagram and another diagram that is $-$ adequate is called *weakly adequate*.

For example, knot $11n_{146} 9^* - 2 : . - 2$ has $-$ adequate 11-crossing diagram and $+$ adequate 12-crossing diagram $6^* - 2.2. - 2.2.2.0. - 2.0$. Another such example is Perko's knot $10_{161} 3 : -2.0 : -2.0$ (Fig. 4) [Stoi].

A crossing in a link diagram for which there exists a circle in the projection plane intersecting the diagram transversely at that crossing, but not intersecting the diagram at any other point is called *nugatory* crossing. A link diagram is called *reduced* if it has no nugatory crossings. The following theorem holds for reduced alternating link diagrams:

Theorem 1. A reduced alternating diagram is adequate [LiThi,Li,Cro].

Hence, all alternating links are adequate.

Theorem 2. An adequate diagram has minimal crossing number [LiThi,Li,Cro].

This theorem can be used to prove minimality of some non-alternating link diagrams.

Theorem 3. Every unlink diagram is inadequate. Semi-adequate link diagrams are non-trivial [Thist].

A non-minimal diagram of an adequate link can be semi-adequate or inadequate. For example, non-minimal diagram $3\,2\,4 - 2\,2$ of the alternating knot $3\,3\,2\,3$ is semi-adequate, and non-minimal diagram $3\,3\,4 - 1\,2$ of the alternating knot $3\,3\,2$ is inadequate.

A non-minimal diagram of a semi-adequate link also can be semi-adequate or inadequate. For example, non-minimal diagram $3, 3, 2, 2 - 3$ and minimal diagram of the same knot $3, 3, 2, -2 - 2$ are both semi-adequate; minimal diagram of the knot $2\,1, 3, -2$ is semi-adequate, and its non-minimal diagram $2\,1, 3, 2-$ is inadequate.

Theorem 4. Two adequate diagrams of a link have the same crossing number and the same writhe [Cro].

Definition 4. An alternating diagram of a marked 2-tangle t is called *strongly alternating* if the both its closures, numerator closure $N(t)$ and denominator closure $D(t)$, are irreducible [LiThi, Li, Cro].

Theorem 5. The non-alternating sum of two strongly alternating tangles is adequate [LiThi, Li, Cro].

This theorem can be very efficiently used to prove that certain types of link diagrams are adequate. For example, all semi-alternating diagrams are adequate [LiThi, Li].

According to Theorem 2, minimal diagrams can be used to determine if a link is adequate, but do not provide necessary and sufficient conditions to distinguish semi-adequate links from inadequate ones.

Theorem 6. A link is inadequate if both coefficients of the terms of highest and lowest degree of its Jones polynomial are different from ± 1 .

The proof of this theorem for knots follows directly from the results of W.B.R. Lickorish and M. Thistlethwaite, and it also holds for links, due to work of J. Przytycki [LiThi, Pr].

2. Adequate links with at most 12 crossings

Using *Knotscape* tables of knots given in Dowker-Thistlethwaite notation, A. Stojmenow detected all non-alternating adequate knots up to $n = 16$ crossings. In this paper we consider adequacy of non-alternating links and their families (classes) given in Conway notation.

Adequate non-alternating links with $n \leq 10$ crossings are given in the following table:

$n = 8$		
$2, 2, -2, -2$	$(2, 2) - (2, 2)$	
2 Links		
$n = 9$		
$3, 2, -2, -2$	$(3, 2) - (2, 2)$	$(2\ 1, 2) - (2, 2)$
$. - (2, 2)$		
4 Links		
$n = 10$		
$(3, 2) - (3, 2)$	$(3, 2) - (2\ 1, 2)$	$(2\ 1, 2) - (2\ 1, 2)$
3 Knots		
$3, 2\ 1, -2, -2$	$3, 3, -2, -2$	$3, -2, 2\ 1, -2$
$3, -2, 3, -2$	$4, 2, -2, -2$	$2, 2, 2, -2, -2$
$2\ 2, 2, -2, -2$	$(4, 2) - (2, 2)$	$(3, 2\ 1) - (2, 2)$
$(3\ 1, 2) - (2, 2)$	$(2\ 1, 2\ 1) - (2, 2)$	$(3, 3) - (2, 2)$
$(2\ 1\ 1, 2) - (2, 2)$	$(2, -2, -2)(2, 2)$	$(2\ 2, 2) - (2, 2)$
$(2, 2, 2) - (2, 2)$	$(2, 2), 2, -(2, 2)$	$. - (2, 2).2$
$. - (2, 2).2\ 0$	$. - (2, 2) : 2\ 0$	$. - (2, 2) : 2$
$10\ 3^* - 1. - 1. - 1. - 1 :: . - 1$		
22 Links		

All of them, except polyhedral ones, satisfy Theorem 5 or are obtained from the pretzel links which satisfy this theorem by permuting their rational tangles.

Theorem 6 gives sufficient but not necessary conditions for recognizing inadequate links. For example, the first and last coefficient of Jones polynomial of the knot $11n_{95} = 20. - 2\ 1. - 2\ 0.2$ are different from ± 1 , so it is inadequate [Cro]. However, since this theorem does not give necessary conditions for a link to be inadequate, the main problem remains detection of inadequate links.

For knots with at most $n \leq 12$ crossings every minimal diagram of a semi-adequate knot is semi-adequate. Unfortunately, this is not true for knots with $n \geq 13$ crossings: the first example of a semi-adequate knot with a minimal inadequate diagram (Fig. 2) is the knot $13n_{4084} 10^{**}. - 1. - 1. - 1 : . - 2.2. - 2$ with the minimal Dowker-Thistlethwaite code

$$\{\{13\}, \{6, -10, 12, 24, 20, -18, -26, -22, -4, 2, -16, 8, -14\}\}.$$

Except this inadequate diagram of writhe 9, it has another semi-adequate minimal diagram $11^{**}. - 2 :: -2\ 0 : -1. - 1. - 1$ of writhe 7, with the Dowker-Thistlethwaite code

$$\{\{13\}, \{6, 12, -16, 23, 2, 17, 21, 26, 11, -4, -25, 7, 13\}\}$$

[KidSto, Stoi2]. For $n = 15$ appear first semi-adequate knots without a minimal semi-adequate diagram. For example, knot $15n_{164563}$ has only minimal diagram $10^{**} - 1. - 2\ 0.2\ 0 :: .2\ 0.2\ 0. - 2\ 0$, and it is inadequate (Fig. 3). However, it has 16-crossing diagram $11^*2\ 0. - 1. - 2. - 1.3\ 0. - 1.2\ 0 :: -1$ which is semi-adequate [Stoi3]. This example can be generalized to the family of knot diagrams $10^{**} - 1. - 2\ 0.(2k)\ 0 :: .2\ 0.2\ 0. - 2\ 0$ and $11^*(2k)\ 0. - 1. - 2. - 1.3\ 0. - 1.2\ 0 :: -1$ ($k \geq 1$) with the same properties, respectively.

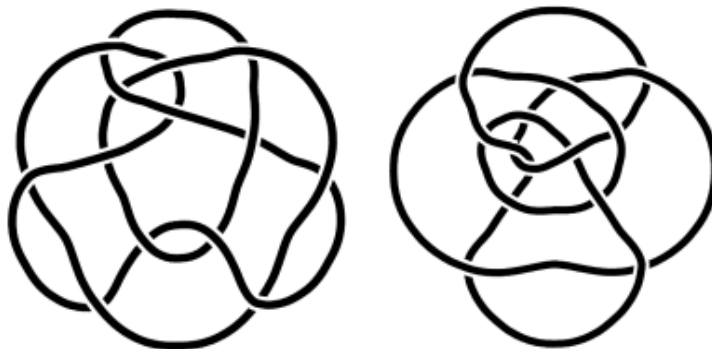


Figure 2: Semi-adequate knot $13n_{4084}$ with a minimal inadequate diagram $10^{**} \cdot -1 \cdot -1 \cdot -1 : -2 \cdot 2 \cdot -2$ and minimal semi-adequate diagram $11^{**} \cdot -2 :: -20 : -1 \cdot -1 \cdot -1$ [KidSto,Stoi2].

For knots with at most $n \leq 12$ crossings we checked adequacy using all their minimal diagrams, but for all links and knots with $n \geq 13$ crossings for each link or knot we used only one minimal diagram.

The sign of adequacy is not necessarily the same for all minimal diagrams of the same link, so we obtain weakly adequate links.

An example of a weakly adequate knot is Perko pair $6^*3 : -20 : -20$ and $6^* - 2 - 1 \cdot -1 \cdot 2 \cdot 0 \cdot -1 \cdot 2 \cdot 0 \cdot -1$ [Stoi]. This example generalizes to one-parameter knot families called *Perko families* [JaSaz]. Conway symbols $6^*(2k+1) : -20 : -20$ and $6^* - (2k) - 1 \cdot -1 \cdot 2 \cdot 0 \cdot -1 \cdot 2 \cdot 0 \cdot -1$ represent two families of minimal diagrams of the same weakly adequate knots with adequacy of different signs and different writhe. For $k = 1$ we obtain Perko pair (Fig. 4), for $k = 2$ two diagrams of the knot $12n_{850}$, for $k = 3$ two diagrams of the knot $14n_{26229}$, and for $k = 4$ two diagrams of the knot $16n_{965076}$ given in *Knotscape* notation. The same holds for the minimal diagrams $6^*2(2k) : -20 : -20$ and $6^* - 2 - (2k-1) - 1 \cdot -1 \cdot 2 \cdot 0 \cdot -1 \cdot 2 \cdot 0 \cdot -1$ of the knots $11n_{135}$, $13n_{3546}$, and $15n_{114094}$ obtained for $k = 1, 2, 3$, respectively. Hence, for every $n \geq 10$ there exists at least one weakly adequate knot which has two minimal diagrams with adequacy of different signs and different writhe. Moreover, if t is any positive rational tangle ($t \neq 1$)*, minimal diagrams $6^*t(k+1) : -20 : -20$ and $6^*(-t)(-k)(-1) \cdot -1 \cdot 2 \cdot 0 \cdot -1 \cdot 2 \cdot 0 \cdot -1$ of the same link have adequacy of different signs and different writhe. Two minimal diagrams of the knot obtained for $t = 22$ and $k = 3$ are illustrated in Fig. 5. In all these cases, the writhe of the diagrams differs by 2, the first diagram is +adequate, and the other -adequate. Since the class $6^*t(k+1) : -20 : -20$ contains links as well (e.g., for $t = 21$, $k = 2$), this is the first example of weakly adequate links.

*A rational tangle is called positive if its Conway symbol contains only positive numbers, and negative if it contains only negative numbers.

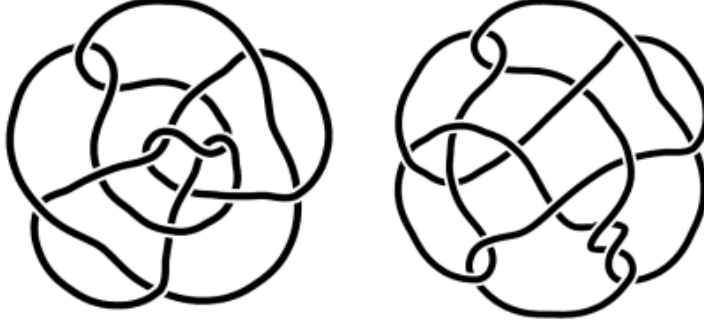


Figure 3: Semi-adequate knot $15n_{164563}$ which has only minimal diagram $10^{**} - 1. - 20.20 :: .20.20. - 20$ which is inadequate and non-minimal 16-crossing diagram $11*20. - 1. - 2. - 1.30. - 1.20 :: -1$ which is semi-adequate [Stoi3].

At least for small number of crossings, most of non-alternating links are semi-adequate, so adequate and inadequate links represent a small portion of all non-alternating links. Hence, it is of interest to tabulate adequate non-alternating links and candidates for inadequate links and try to find some general criteria for adequacy of certain classes of links. We checked adequacy of all minimal diagrams of non-alternating knots and links with at most $n = 12$ crossings given in Conway notation.

Among 202 non-alternating links with at most 10 crossings there are only 28 adequate links and 3 adequate knots. Links with inadequate minimal diagrams are even more rare. Their list for $n = 10$ is given in the following table:

$n = 10$		
$(2, 2, -2) (2, -2)$	$2. - 20. - 2.20$	$103^* - 1. - 1 :: -1. - 1$
3 Links		

where links $2. - 20. - 2.20$ and $103^* - 1. - 1 :: -1. - 1$ are inadequate according to Theorem 6, and nothing is known for the link $(2, 2, -2) (2, -2)$.

Particular links, families or classes of links which have a minimal inadequate diagram will be referred to us as *candidates for inadequate links* and in some cases Theorem 6 will confirm that they indeed are inadequate.

Candidates for inadequate knots occur for the first time among 11-crossing knots: knot $20. - 21. - 20.2$ is inadequate according to Theorem 6, but for the knot $20. - 3. - 20.2$ which all minimal diagrams are inadequate, it is not possible to make any conclusion, since both leading coefficients of its Jones polynomial are equal to 1.

For $n = 12$, among 19 knots with an inadequate minimal diagram, 11 knots given in the following table are inadequate according to Theorem 6

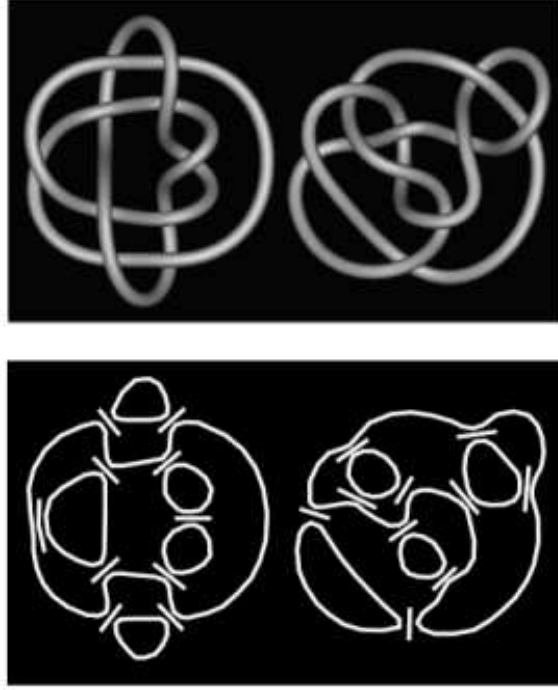


Figure 4: Perko pair: semi-adequate knot with two minimal diagrams $6^*3 : -20 : -20$ and $6^* - 2 - 1. - 1.20. - 1.20. - 1$ with the adequacy of different signs, where the first is +adequate, and the other -adequate.

$2. - 20. - 2.2110$	$2 : (-2, 21) 0 : -20$	$2 : (2, -2 - 1) 0 : -20$
$2.2. - 2.20. - 2 - 1$	$3. - 20. - 2.210$	$3. - 2 - 10. - 2.20$
$8^*20. - 20. - 20.20$	$8^* - 2 - 1.20. - 2$	$9^*. - 2 : -20. - 2$
$101^* - 20 :: . - 20$	$102^* - 20 :: -2$	

while inadequacy of the remaining 8 knots from the following table remains unknown

$2. - 30. - 2 - 1.20$	$8^* - 2 - 1 - 1 :: -20$	$8^*2 : . - 20 : . - 2 - 10$
$8^* - 2 - 1 :: -30$	$8^*2 : . - 2 - 10 : . - 20$	$8^* - 20.2 : -2 - 10$
$8^* - 2.2. - 20 : 20$	$8^* - 20 : -20 : -20 : 20$	

For $n = 11$ four links

$(21, 2) - 1 - 1(2, 2)$	$(2, 2), -2, -1, (2, -2)$	$6^*3. - 20. - 2.20$	$6^*(2, -2).2. - 2$
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are inadequate according to Theorem 6, and the following 8 links are candidates for inadequate links. All their minimal diagrams are inadequate.

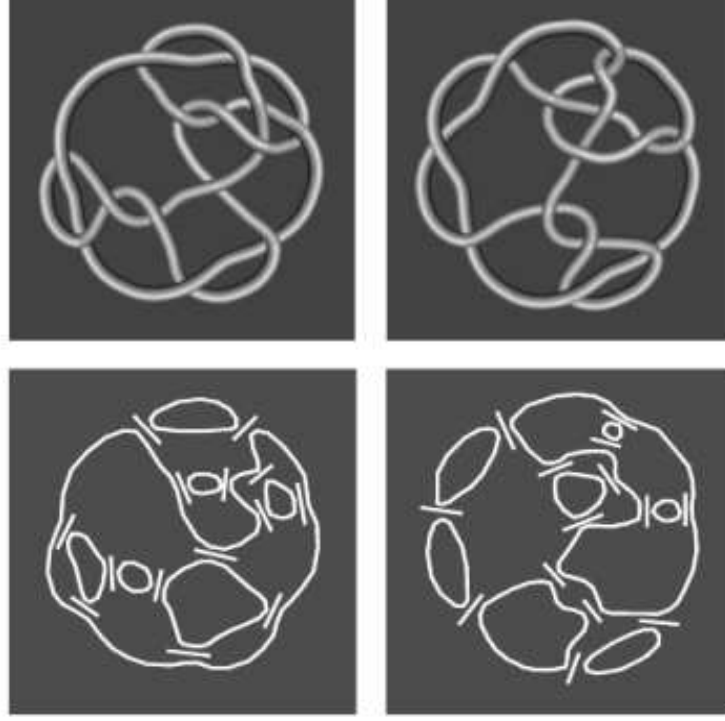


Figure 5: Perko-type pair of knot diagrams: semi-adequate knot with two minimal diagrams $6^*223 : -20 : -20$ and $6^* - 2 - 2 - 2 - 1. - 1.20. - 1.20. - 1$ with the adequacy of different signs, where the first is $+$ -adequate, and the other $-$ -adequate.

$(-2 - 1, 2) 11 (2, -2)$	$(-2 - 1, 2, 2) (2, -2)$	$(21, 2, -2) (2, -2)$	$(2, 2) - 1 - 1 - 1 (2, -2)$
$(2, 2, -2) (-2 - 1, 2)$	$(2, 2, -2) (21, -2)$	$6^*21. - 20. - 2.20$	$6^*(2, -2), -2$

For $n = 12$ inadequacy of 63 links is confirmed according to Theorem 6, and the remaining 232 links are candidates for inadequate links.

Tables of adequate non-alternating links with at most $n = 12$ crossings in Conway notation can be downloaded in the form of *Mathematica* notebook from the address:

<http://www.mi.sanu.ac.yu/vismath/adequate.pdf>

3. Families and classes of links and their adequacy

Definition 5. For a link L given in an unreduced[†] Conway notation $C(L)$, let S

[†]The Conway notation is called unreduced if in symbols of polyhedral links elementary tangles 1 in single vertices are not omitted.

denote a set of numbers in the Conway symbol, excluding numbers denoting basic polyhedra and zeros (marking the position of tangles in the vertices of polyhedra). For $C(L)$ and an arbitrary (non-empty) subset \tilde{S} of S the family $F_{\tilde{S}}(L)$ of knots or links derived from L is constructed by substituting each $a \in S_f$, $a \neq 1$, by $\text{sgn}(a)(|a| + k_a)$ for $k_a \in N$ [JaSaz].

If k_a is an even number ($k_a \in N$), the number of components is preserved inside a family, i.e., we obtain families of knots or links with the same number of components.

Definition 6. A link given by Conway symbol containing only tangles 1, -1 , 2, or -2 is called a *source link*. A link given by Conway symbol containing only tangles 1, -1 , 2, -2 , 3, or -3 is called a *generating link*.

Theorem 7. All link diagrams which belong to the same family of diagrams have adequacy of the same sign.

Proof: If we substitute $a \in S_f$, $a \neq 1$, by $\text{sgn}(a)(|a| + 1)$ (Definition 5), a new state circle of the length 2 appears in one of the states D_{s_+} or D_{s_-} , so the sign of adequacy remains unchanged. In the remaining state the number of state circles remains unchanged and all state circles associated with the new crossing obtain one new touching point. If the crossings of the original tangle a after smoothing correspond to different state circles, the same holds for the tangle $\text{sgn}(a)(|a| + 1)$, and the sign of adequacy remains unchanged. By induction, we conclude that this property holds for every $k_a \in N^{\ddagger}$. Hence, all link diagrams which belong to the same family of diagrams have the adequacy of the same sign. \square

Proposition 1. The adequacy of a link diagram remains unchanged if we replace every positive rational tangle by 2, and every negative rational tangle by -2 .

The proof of this Proposition is straightforward, because every rational alternating tangle is adequate, so its collapse into a bigon does not change the sign of adequacy.

A pretzel tangle and the pretzel link obtained as its closure, consisting from n alternating rational tangles t_i is denoted by t_1, t_2, \dots, t_n ($n \geq 3$, $t_i \neq 1$, $i = 1, \dots, n$). Number n will be called the *length of the pretzel tangle*.

Theorem 8. A non-alternating pretzel link t_1, t_2, \dots, t_n is semi-adequate if it contains exactly one rational tangle of one sign, and all the other rational tangles of the opposite sign. Otherwise, it is adequate.

A pretzel tangle is called adequate or semi-adequate if its corresponding pretzel link is adequate or semi-adequate, respectively.

Let's denote source link of the form $2, \dots, -2, \dots$, where 2 occurs k times, and -2 occurs l times with the short symbol $(2)^k, (-2)^l$. For different values of k and l we obtain six classes of source links, where all members of the same class have the adequacy of the same sign

[‡]See Def. 5.

$k \geq 3, l = 0$	+alternating
$k = 0, l \geq 3$	−alternating
$k \geq 2, l \geq 2$	adequate
$k = 1, l \geq 2$	+adequate
$k \geq 2, l = 1$	−adequate
$k = l = 1$	inadequate

This property directly follows from Theorem 1 and Theorem 8. As the minimal representatives of these six classes we can use source links $(2, 2, 2)$, $(-2, -2, -2)$, $(2, 2, -2, -2)$, $(-2, -2, 2)$, $(2, 2, -2)$, and $(2, -2)$, respectively. Combining this with Proposition 1 we conclude that these six source links can be used as the representatives of the corresponding pretzel links (Montesinos links) with the rational tangles of the corresponding signs. For example, source link $2, 2, -2, -2$ can be used as the representative of all non-alternating adequate pretzel links of the form $t_1, \dots, t_k, -t'_1, \dots, -t'_l$, $(k \geq 2, l \geq 2)$, where t_i ($i = 1, 2, \dots, k$) and t'_j ($j = 1, 2, \dots, l$) are positive rational tangles different from 1.

4. Some particular classes of algebraic links and their adequacy

Definition 7. An alternating pretzel tangle $P_n = t_1, t_2, \dots, t_n$ is called +alternating if all its rational tangles t_i are positive, and −alternating if they are all negative.

Tangle $t_1, -t_2$ is inadequate, where t_1, t_2 are positive rational tangles.

Theorem 9. A link $P_m Q_n = (p_1, p_2, \dots, p_m) (q_1, q_2, \dots, q_n)$ ($m, n \geq 2$) obtained as the product of pretzel tangles P_m and Q_n is adequate if

- both P_m and Q_n are adequate; or
- one of them is +alternating, and the other +adequate; or
- one of them is −alternating, and the other −adequate.

It is semi-adequate if

- one of them is adequate, and the other semi-adequate; or
- one of them is +adequate, and the other −adequate; or
- if one of them is inadequate, and the other an alternating pretzel tangle.

It is candidate for inadequate if

- both P_m and Q_n are +adequate or −adequate;
- if one of them is inadequate, and the other is not an alternating pretzel tangle.

From the preceding theorem we obtain the following multiplication table, where the * denotes the product of pretzel tangles[§]:

[§]The product $P_1 P_2$ of inadequate tangles P_1 and P_2 is omitted, since it represents a non-minimal diagram of an alternating link.

*	+alt	-alt	adq	+adq	-adq	inadeq
+alt	+alt	adq	adq	adq	+adq	+adq
-alt	adq	-alt	adq	-adq	adq	-adq
adq	adq	adq	adq	-adq	+adq	inadeq
+adq	adq	+adq	+adq	inadeq	+adq	inadeq
-adq	-adq	adq	-adq	-adq	inadeq	inadeq
inadeq	-adq	+adq	inadeq	inadeq	inadeq	

For links of the form $P_m Q_n = (p_1, p_2, \dots, p_n) (q_1, q_2, \dots, q_n)$ we obtained general rules for adequacy, expressed as the multiplication table. Unfortunately, for links of the form $P_1 P_2 \dots P_k$, with $k \geq 2$ we are not able to present general adequacy multiplication tables.

As the minimal representatives of pretzel tangles with the properties **+alt**, **-alt**, **adq**, **+adq**, **-adq**, and **inadeq** we can use the following tangles:

1	+alt	2, 2, 2
2	-alt	-2, -2, -2
3	adq	2, 2, -2, -2
4	+adq	-2, -2, 2
5	-adq	2, 2, -2
6	inadeq	2, -2

If we denote the properties **+alt**, **-alt**, **adq**, **+adq**, **-adq**, and **inadeq** by 1-6, for $k = 3$, we have the following statement:

Theorem 10. The links $P_1 P_2 P_3$ are adequate for the following properties of pretzel tangles P_1, P_2, P_3 :

1, 1, 1	1, 1, 2	1, 1, 3	1, 1, 4	1, 2, 1	1, 2, 2	1, 2, 3	1, 3, 1
1, 3, 2	1, 3, 3	1, 4, 1	1, 4, 2	1, 4, 3	1, 5, 2	1, 6, 2	2, 1, 2
2, 1, 3	2, 2, 2	2, 2, 3	2, 2, 5	2, 3, 2	2, 3, 3	2, 5, 2	2, 5, 3
3, 1, 3	3, 1, 4	3, 2, 3	3, 2, 5	3, 3, 3	4, 1, 4	5, 2, 5	

semi-adequate for:

1, 1, 5	1, 1, 6	1, 2, 4	1, 2, 5	1, 3, 4	1, 3, 5	1, 4, 4	1, 4, 5
1, 5, 1	1, 5, 3	1, 5, 4	1, 6, 1	1, 6, 3	1, 6, 4	2, 1, 4	2, 1, 5
2, 2, 4	2, 2, 6	2, 3, 4	2, 3, 5	2, 4, 2	2, 4, 3	2, 4, 5	2, 5, 4
2, 5, 5	2, 6, 2	2, 6, 3	2, 6, 5	3, 1, 5	3, 1, 6	3, 2, 4	3, 2, 6
3, 3, 4	3, 3, 5	3, 4, 3	3, 4, 5	3, 5, 3	3, 5, 4	4, 1, 5	4, 1, 6
4, 2, 4	4, 2, 5	4, 2, 6	4, 3, 4	4, 5, 4	5, 1, 5	5, 1, 6	5, 2, 6
5, 3, 5	5, 4, 5	6, 1, 6	6, 2, 6				

and candidates for inadequate for:

1, 2, 6	1, 3, 6	1, 4, 6	1, 5, 5	1, 5, 6	1, 6, 5	1, 6, 6	2, 1, 6
2, 3, 6	2, 4, 4	2, 4, 6	2, 5, 6	2, 6, 4	2, 6, 6	3, 3, 6	3, 4, 4
3, 4, 6	3, 5, 5	3, 5, 6	3, 6, 3	3, 6, 4	3, 6, 5	3, 6, 6	4, 3, 5
4, 3, 6	4, 4, 4	4, 4, 5	4, 4, 6	4, 5, 5	4, 5, 6	4, 6, 4	4, 6, 5
4, 6, 6	5, 3, 6	5, 4, 6	5, 5, 5	5, 5, 6	5, 6, 5	5, 6, 6	6, 3, 6
6, 4, 6	6, 5, 6	6, 6, 6					

The results hold for all sequences a, b, c ($a, b, c \in \{1, 2, \dots, 6\}$) and their reverses. Analogous tables are obtained by computer calculations for all $k \leq 6$.

For a given non-alternating pretzel tangle P the tangle P' obtained from it by replacing every rational positive or negative tangle t_i with the tangle $\text{sign}(t_i) \times 2$ will be called basic pretzel tangle.

Theorem 11. The links $P_1 P_2 \dots P_k$ and $P'_1 P'_2 \dots P'_k$ have the same adequacy.

Next, we will consider links of the form P_1, P_2, \dots, P_k ($k \geq 3$, where P_i ($i = 1, \dots, k$) are pretzel tangles. Since permutation of pretzel tangles preserves the sign of adequacy, the result holds for every sequence a, b, c ($a, b, c \in \{1, 2, \dots, 6\}$) and all of its permutations. For $k = 3$ we obtained the following result:

Theorem 12. The links P_1, P_2, P_3 are adequate for the following properties of pretzel tangles P_1, P_2, P_3 :

1, 1, 1	1, 1, 2	1, 1, 3	1, 1, 4	1, 2, 2	1, 2, 3	1, 3, 3	1, 3, 4
1, 4, 4	2, 2, 2	2, 2, 3	2, 2, 5	2, 3, 3	2, 3, 5	2, 5, 5	3, 3, 3
3, 3, 4	3, 3, 5	3, 3, 6	3, 4, 4	3, 4, 5	3, 4, 6	3, 5, 5	3, 5, 6
3, 6, 6	4, 4, 4	4, 4, 5	4, 4, 6	4, 5, 5	4, 5, 6	4, 6, 6	5, 5, 5
5, 5, 6	5, 6, 6	6, 6, 6					

semi-adequate for:

1, 2, 4	2, 2, 4	2, 2, 6	2, 3, 4	2, 3, 6	2, 4, 4	2, 4, 5	2, 4, 6
2, 5, 6	2, 6, 6	1, 1, 5	1, 1, 6	1, 2, 5	1, 3, 5	1, 3, 6	1, 4, 5
1, 4, 6	1, 5, 5	1, 5, 6	1, 6, 6				

and candidates for inadequate for:

1, 2, 6

Analogous results are obtained by computer calculations for all $k \leq 6$.

Next we consider links of the form $(P_1, P_2, \dots, P_m) (Q_1, Q_2, \dots, Q_n)$ ($m, n \geq 2$), where P_i and Q_j ($i = 1, \dots, m, j = 1, \dots, n$) are pretzel tangles.

In the case $m = n = 2$, where in the sequence $(a, b) (c, d)$ ($a, b, c, d \in \{1, 2, \dots, 6\}$), a and b , and c and d can commute and all sequences $(a, b) (c, d)$ can be reversed, we obtain the following result:

Theorem 13. The links $(P_1, P_2) (Q_1, Q_2)$ are adequate for the following properties of pretzel tangles P_1, P_2, Q_1, Q_2 :

(1, 1) (1, 1)	(1, 1) (1, 2)	(1, 1) (1, 3)	(1, 1) (1, 4)	(1, 1) (2, 2)	(1, 1) (2, 3)	(1, 1) (3, 3)
(1, 1) (3, 4)	(1, 1) (4, 4)	(1, 2) (1, 2)	(1, 2) (1, 3)	(1, 2) (1, 4)	(1, 2) (2, 2)	(1, 2) (2, 3)
(1, 2) (2, 5)	(1, 2) (3, 3)	(1, 2) (3, 4)	(1, 2) (3, 5)	(1, 2) (3, 6)	(1, 2) (4, 4)	(1, 2) (4, 5)
(1, 2) (4, 6)	(1, 2) (5, 5)	(1, 2) (5, 6)	(1, 2) (6, 6)	(1, 3) (1, 3)	(1, 3) (1, 4)	(1, 3) (2, 2)
(1, 3) (2, 3)	(1, 3) (3, 3)	(1, 3) (3, 4)	(1, 3) (4, 4)	(1, 4) (1, 4)	(2, 2) (2, 2)	(2, 2) (2, 3)
(2, 2) (2, 5)	(2, 2) (3, 3)	(2, 2) (3, 5)	(2, 2) (5, 5)	(2, 3) (2, 3)	(2, 3) (2, 5)	(2, 3) (3, 3)
(2, 3) (3, 5)	(2, 3) (5, 5)	(2, 5) (2, 5)	(3, 3) (3, 3)			

semi-adequate for:

(1, 1) (1, 5)	(1, 1) (1, 6)	(1, 1) (2, 5)	(1, 1) (3, 5)	(1, 1) (3, 6)	(1, 1) (4, 5)	(1, 1) (4, 6)
(1, 1) (5, 5)	(1, 1) (5, 6)	(1, 1) (6, 6)	(1, 2) (1, 5)	(1, 2) (1, 6)	(1, 3) (1, 5)	(1, 3) (1, 6)
(1, 3) (2, 5)	(1, 3) (3, 5)	(1, 3) (3, 6)	(1, 3) (4, 5)	(1, 3) (4, 6)	(1, 3) (5, 5)	(1, 3) (5, 6)
(1, 3) (6, 6)	(1, 4) (1, 5)	(1, 4) (1, 6)	(1, 4) (2, 2)	(1, 4) (2, 3)	(1, 4) (2, 5)	(1, 4) (3, 3)
(1, 4) (3, 4)	(1, 4) (3, 5)	(1, 4) (3, 6)	(1, 4) (4, 4)	(1, 4) (4, 5)	(1, 4) (4, 6)	(1, 4) (5, 5)
(1, 4) (5, 6)	(1, 4) (6, 6)	(2, 4) (2, 5)	(2, 4) (3, 3)	(2, 4) (3, 5)	(2, 4) (5, 5)	(3, 3) (3, 5)
(3, 3) (5, 5)	(3, 4) (3, 5)	(3, 4) (5, 5)	(4, 4) (5, 5)	(1, 1) (2, 4)	(1, 2) (2, 4)	(1, 2) (2, 6)
(1, 3) (2, 4)	(1, 5) (2, 2)	(1, 5) (2, 3)	(1, 5) (2, 4)	(1, 5) (3, 3)	(1, 5) (3, 4)	(1, 5) (4, 4)
(2, 2) (2, 4)	(2, 2) (2, 6)	(2, 2) (3, 4)	(2, 2) (3, 6)	(2, 2) (4, 4)	(2, 2) (4, 5)	(2, 2) (4, 6)
(2, 2) (5, 6)	(2, 2) (6, 6)	(2, 3) (2, 4)	(2, 3) (2, 6)	(2, 3) (3, 4)	(2, 3) (3, 6)	(2, 3) (4, 4)
(2, 3) (4, 5)	(2, 3) (4, 6)	(2, 3) (5, 6)	(2, 3) (6, 6)	(2, 5) (2, 6)	(2, 5) (3, 3)	(2, 5) (3, 4)
(2, 5) (3, 5)	(2, 5) (3, 6)	(2, 5) (4, 4)	(2, 5) (4, 5)	(2, 5) (4, 6)	(2, 5) (5, 5)	(2, 5) (5, 6)
(2, 5) (6, 6)	(3, 3) (3, 4)	(3, 3) (4, 4)	(3, 5) (4, 4)			

and candidates for inadequate for:

(1, 1) (2, 6)	(1, 3) (2, 6)	(1, 4) (2, 4)	(1, 4) (2, 6)	(1, 5) (1, 5)	(1, 5) (1, 6)	(1, 5) (2, 5)
(1, 5) (2, 6)	(1, 5) (3, 5)	(1, 5) (3, 6)	(1, 5) (4, 5)	(1, 5) (4, 6)	(1, 5) (5, 5)	(1, 5) (5, 6)
(1, 5) (6, 6)	(1, 6) (1, 6)	(1, 6) (2, 2)	(1, 6) (2, 3)	(1, 6) (2, 4)	(1, 6) (2, 5)	(1, 6) (2, 6)
(1, 6) (3, 3)	(1, 6) (3, 4)	(1, 6) (3, 5)	(1, 6) (3, 6)	(1, 6) (4, 4)	(1, 6) (4, 5)	(1, 6) (4, 6)
(1, 6) (5, 5)	(1, 6) (5, 6)	(1, 6) (6, 6)	(2, 4) (2, 4)	(2, 4) (2, 6)	(2, 4) (3, 4)	(2, 4) (3, 6)
(2, 4) (4, 4)	(2, 4) (4, 5)	(2, 4) (4, 6)	(2, 4) (5, 6)	(2, 4) (6, 6)	(2, 6) (2, 6)	(2, 6) (3, 3)
(2, 6) (3, 4)	(2, 6) (3, 5)	(2, 6) (3, 6)	(2, 6) (4, 4)	(2, 6) (4, 5)	(2, 6) (4, 6)	(2, 6) (5, 5)
(2, 6) (5, 6)	(2, 6) (6, 6)	(3, 3) (3, 6)	(3, 3) (4, 5)	(3, 3) (4, 6)	(3, 3) (5, 6)	(3, 3) (6, 6)
(3, 4) (3, 4)	(3, 4) (3, 6)	(3, 4) (4, 4)	(3, 4) (4, 5)	(3, 4) (4, 6)	(3, 4) (5, 6)	(3, 4) (6, 6)
(3, 5) (3, 5)	(3, 5) (3, 6)	(3, 5) (4, 5)	(3, 5) (4, 6)	(3, 5) (5, 5)	(3, 5) (5, 6)	(3, 5) (6, 6)
(3, 6) (3, 6)	(3, 6) (4, 4)	(3, 6) (4, 5)	(3, 6) (4, 6)	(3, 6) (5, 5)	(3, 6) (5, 6)	(3, 6) (6, 6)
(4, 4) (4, 4)	(4, 4) (4, 5)	(4, 4) (4, 6)	(4, 4) (5, 6)	(4, 4) (6, 6)	(4, 5) (4, 5)	(4, 5) (4, 6)
(4, 5) (5, 5)	(4, 5) (5, 6)	(4, 5) (6, 6)	(4, 6) (4, 6)	(4, 6) (5, 5)	(4, 6) (5, 6)	(4, 6) (6, 6)
(5, 5) (5, 5)	(5, 5) (5, 6)	(5, 5) (6, 6)	(5, 6) (5, 6)	(5, 6) (6, 6)	(6, 6) (6, 6)	

Analogous results are obtained by computer calculations for $m, n \leq 4$.

Furthermore we consider links of the form $P_1, t_1, t_2, \dots, t_n$, where P_1 is a pretzel tangle, and t_i ($i = 1, 2, \dots, n, n \geq 2$) are rational tangles. If $P = t_1, t_2, \dots, t_n$, we have the following statement:

- links of the given form are adequate if $\{P_1, P\} \in \{\{1, 1\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 2\}, \{2, 3\}, \{2, 5\}, \{3, 3\}\}$;
- semi-adequate if $\{P_1, P\} \in \{\{1, 5\}, \{1, 6\}, \{2, 4\}, \{2, 6\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}$;
- and candidates for inadequate if $\{P_1, P\} \in \{\{3, 6\}, \{4, 4\}, \{4, 6\}, \{5, 5\}, \{5, 6\}, \{6, 6\}\}$.

Next we consider links of the form $P_1, \dots, P_m, t_1, t_2, \dots, t_n$, where P_1, \dots, P_m are pretzel tangles, and t_i ($i = 1, 2, \dots, n, n \geq 2$) are rational tangles. If $P = t_1, t_2, \dots, t_n$, for $m = 2$ we have the following statement:

Links of the given form are adequate if $(\{P_1, P_2\}, P)$ is:

(\{1, 1\}, 1)	(\{1, 1\}, 2)	(\{1, 1\}, 3)	(\{1, 1\}, 4)	(\{1, 2\}, 1)	(\{1, 2\}, 2)
(\{1, 2\}, 3)	(\{1, 2\}, 4)	(\{1, 2\}, 5)	(\{1, 2\}, 6)	(\{1, 3\}, 1)	(\{1, 3\}, 2)
(\{1, 3\}, 3)	(\{1, 3\}, 4)	(\{1, 4\}, 1)	(\{2, 2\}, 1)	(\{2, 2\}, 2)	(\{2, 2\}, 3)
(\{2, 2\}, 5)	(\{2, 3\}, 1)	(\{2, 3\}, 2)	(\{2, 3\}, 3)	(\{2, 3\}, 5)	(\{2, 5\}, 2)
(\{3, 3\}, 1)	(\{3, 3\}, 2)	(\{3, 3\}, 3)	(\{3, 4\}, 1)	(\{3, 5\}, 2)	(\{4, 4\}, 1)
(\{5, 5\}, 2)					

semi-adequate if $(\{P_1, P_2\}, P)$ is:

$(\{1, 1\}, 5)$	$(\{1, 1\}, 6)$	$(\{1, 3\}, 5)$	$(\{1, 3\}, 6)$	$(\{1, 4\}, 2)$	$(\{1, 4\}, 3)$
$(\{1, 4\}, 4)$	$(\{1, 4\}, 5)$	$(\{1, 4\}, 6)$	$(\{1, 5\}, 1)$	$(\{1, 5\}, 2)$	$(\{1, 5\}, 3)$
$(\{1, 5\}, 4)$	$(\{1, 6\}, 1)$	$(\{2, 2\}, 4)$	$(\{2, 2\}, 6)$	$(\{2, 3\}, 4)$	$(\{2, 3\}, 6)$
$(\{2, 4\}, 1)$	$(\{2, 4\}, 2)$	$(\{2, 4\}, 3)$	$(\{2, 4\}, 5)$	$(\{2, 5\}, 1)$	$(\{2, 5\}, 3)$
$(\{2, 5\}, 4)$	$(\{2, 5\}, 5)$	$(\{2, 5\}, 6)$	$(\{2, 6\}, 2)$	$(\{3, 3\}, 4)$	$(\{3, 3\}, 5)$
$(\{3, 4\}, 2)$	$(\{3, 4\}, 3)$	$(\{3, 4\}, 5)$	$(\{3, 5\}, 1)$	$(\{3, 5\}, 3)$	$(\{3, 5\}, 4)$
$(\{3, 6\}, 1)$	$(\{3, 6\}, 2)$	$(\{4, 4\}, 2)$	$(\{4, 4\}, 3)$	$(\{4, 4\}, 5)$	$(\{4, 5\}, 1)$
$(\{4, 5\}, 2)$	$(\{4, 6\}, 1)$	$(\{4, 6\}, 2)$	$(\{5, 5\}, 1)$	$(\{5, 5\}, 3)$	$(\{5, 5\}, 4)$
$(\{5, 6\}, 1)$	$(\{5, 6\}, 2)$	$(\{6, 6\}, 1)$	$(\{6, 6\}, 2)$		

and candidates for inadequate if $(\{P_1, P_2\}, P)$ is:

$(\{1, 5\}, 5)$	$(\{1, 5\}, 6)$	$(\{1, 6\}, 2)$	$(\{1, 6\}, 3)$	$(\{1, 6\}, 4)$	$(\{1, 6\}, 5)$
$(\{1, 6\}, 6)$	$(\{2, 4\}, 4)$	$(\{2, 4\}, 6)$	$(\{2, 6\}, 1)$	$(\{2, 6\}, 3)$	$(\{2, 6\}, 4)$
$(\{2, 6\}, 5)$	$(\{2, 6\}, 6)$	$(\{3, 3\}, 6)$	$(\{3, 4\}, 4)$	$(\{3, 4\}, 6)$	$(\{3, 5\}, 5)$
$(\{3, 5\}, 6)$	$(\{3, 6\}, 3)$	$(\{3, 6\}, 4)$	$(\{3, 6\}, 5)$	$(\{3, 6\}, 6)$	$(\{4, 4\}, 4)$
$(\{4, 4\}, 6)$	$(\{4, 5\}, 3)$	$(\{4, 5\}, 4)$	$(\{4, 5\}, 5)$	$(\{4, 5\}, 6)$	$(\{4, 6\}, 3)$
$(\{4, 6\}, 4)$	$(\{4, 6\}, 5)$	$(\{4, 6\}, 6)$	$(\{5, 5\}, 5)$	$(\{5, 5\}, 6)$	$(\{5, 6\}, 3)$
$(\{5, 6\}, 4)$	$(\{5, 6\}, 5)$	$(\{5, 6\}, 6)$	$(\{6, 6\}, 3)$	$(\{6, 6\}, 4)$	$(\{6, 6\}, 5)$

In the same way, by experimental computer calculations, it is possible to obtain the results for some other types of links. For example, a link of the form $P_1 p P_2$, where P_1, P_2 are pretzel tangles, and p is a positive chain of bigons is adequate if $\{P_1, P_2\} \in \{\{1, 1\}, \{1, 3\}, \{1, 4\}, \{3, 3\}, \{3, 4\}, \{4, 4\}\}$, candidate for inadequate if $\{P_1, P_2\} \in \{\{2, 5\}, \{2, 6\}\}$, and semi-adequate otherwise.

5. Adequacy of polyhedral links

Theorem 14. Polyhedral link with one pretzel tangle P_1 and positive rational tangles in other vertices is adequate if $P_1 \in \{1, 2, 3, 4\}$, and semi-adequate if $P_1 \in \{5, 6\}$.

Theorem 15. In every adequate polyhedral link with two pretzel tangles P_1, P_2 and positive rational tangles in other vertices, $P_1 \notin \{5, 6\}$ and $P_2 \notin \{5, 6\}$.

Condition from the Theorem 15 is necessary, but not sufficient. Hence, we will consider different cases, depending on different polyhedral source links. For example, the following results hold for non-alternating links derived from the basic polyhedron 6^* with two pretzel tangles P_1 and P_2 and positive rational tangles in remaining vertices:

- a link of the form $6^* P_1 P_2 t_1 t_2 t_3 t_4$ is adequate if $\{P_1, P_2\} \in \{\{1, 1\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{3, 3\}, \{3, 4\}, \{4, 4\}\}$, a candidate for inadequate if $\{P_1, P_2\} \in \{\{2, 2\}, \{5, 5\}, \{5, 6\}, \{6, 6\}\}$, and semi-adequate otherwise;
- a link of the form $6^* P_1 P_2 0 t_1 t_2 t_3 t_4$ is adequate if $\{P_1, P_2\} \in \{\{1, 1\}, \{1, 2\}, \{1, 3\}, \{1, 4\}\}$, a candidate for inadequate if $\{P_1, P_2\} \in \{\{2, 2\}\}$, and semi-adequate otherwise;

- a link of the form $6^*P_1.t_1.t_2.P_2.0.t_3.t_4$ is adequate if $P_1 \notin \{5, 6\}$ and $P_2 \notin \{5, 6\}$, and semi-adequate otherwise;
- a link of the form $6^*P_1.t_1.t_2.P_2.t_3.t_4$ is adequate if $P_1 \notin \{5, 6\}$ and $P_2 \notin \{5, 6\}$, and a candidate for inadequate if $\{P_1, P_2\} = \{5, 6\}$.

A link of the form $6^*P_1.P_2.P_3.t_1.t_2.t_3$ is adequate for the following triples (P_1, P_2, P_3) :

1, 1, 1	1, 1, 2	1, 1, 3	1, 1, 4	1, 2, 1	1, 2, 3	1, 2, 4	1, 3, 1
1, 3, 3	1, 3, 4	1, 4, 1	1, 4, 3	1, 4, 4	2, 1, 1	2, 1, 3	2, 1, 4
3, 1, 1	3, 1, 2	3, 1, 3	3, 1, 4	3, 2, 1	3, 2, 3	3, 2, 4	3, 2, 5
3, 2, 6	3, 3, 1	3, 3, 3	3, 3, 4	3, 4, 1	4, 1, 1	4, 1, 2	4, 1, 3
4, 1, 4	4, 2, 1	4, 2, 3	4, 2, 4	4, 2, 5	4, 2, 6	4, 3, 1	4, 3, 3
4, 3, 4	4, 4, 1	5, 2, 3	5, 2, 4	5, 2, 5	5, 2, 6	6, 2, 3	6, 2, 4
6, 2, 5	6, 2, 6						

a candidate for inadequate for:

1, 5, 5	1, 5, 6	1, 6, 5	1, 6, 6	2, 3, 5	2, 3, 6	2, 4, 2	2, 4, 3
2, 4, 4	2, 4, 5	2, 4, 6	2, 5, 5	2, 5, 6	2, 6, 2	2, 6, 3	2, 6, 4
2, 6, 5	2, 6, 6	3, 4, 2	3, 5, 5	3, 5, 6	3, 6, 2	3, 6, 3	3, 6, 4
3, 6, 5	3, 6, 6	4, 4, 2	4, 5, 5	4, 5, 6	4, 6, 2	4, 6, 3	4, 6, 4
4, 6, 5	4, 6, 6	5, 3, 2	5, 4, 2	5, 5, 1	5, 5, 2	5, 5, 3	5, 5, 4
5, 5, 5	5, 5, 6	5, 6, 1	5, 6, 2	5, 6, 3	5, 6, 4	5, 6, 5	5, 6, 6
6, 3, 2	6, 4, 2	6, 5, 1	6, 5, 2	6, 5, 3	6, 5, 4	6, 5, 5	6, 5, 6
6, 6, 1	6, 6, 2	6, 6, 3	6, 6, 4	6, 6, 5	6, 6, 6		

and semi-adequate otherwise.

Except polyhedral links with pretzel tangles, we will consider polyhedral links containing only rational tangles.

Theorem 16. Non-alternating link derived from the basic polyhedron 6^* is a candidate for inadequate if it is obtained from one of the following source links by replacing 2-tangles by positive rational tangles t_i ($i \in \{1, \dots, 6\}$, $t_i \neq 1$)

$6^*2. - 20. - 2.20$	$6^*2.2. - 2.2. - 20$	$6^* - 2.2. - 20.2.2$
$6^*2.2. - 2.20. - 2$	$6^* - 2.20. - 2.20.2$	$6^*2. - 2.2.2.20. - 20$
$6^*2. - 2.20. - 2. - 2. - 20$	$6^*2. - 2. - 2. - 2.2. - 20$	$6^*2. - 2.2.2.2. - 20$
$6^*2. - 2. - 2. - 20.2. - 20$	$6^*2. - 20. - 2. - 20. - 2.20$	$6^*2. - 20. - 2.20. - 2.20$

and semi-adequate otherwise[¶].

[¶]Knot $6^*2. - 2.20. - 2. - 2. - 20$ is recognized as potential inadequate, i.e., as a knot without minimal + or -adequate diagram by M. Thistlethwaite in 1988 [Thi], but to this knot cannot be applied Theorem 6. From 12 links from this table, the five of them: $6^*2. - 20. - 2.20$, $6^*2.2. - 2.2. - 20$, $6^*2.2. - 2.20. - 2$, $6^*2. - 2.2.2.20. - 20$, and $6^*2. - 2. - 2. - 2.2. - 20$ are inadequate, according to Theorem 6.

In the same way, similar results is possible to obtain for other basic polyhedra.

6. Adequacy of mixed states and adequacy number

The definition of adequacy can be extended to an arbitrary state a link diagram D . Together with special states s_+ and s_- , we will consider mixed states, where markers have different signs.

According to Definition 1, a state s of the diagram D is called *adequate state* if, at each crossing, the two segments of D_s which replace the crossing are in different state circles.

Theorem 17. Every link diagram has at least two adequate states.

Proof: Every alternating link diagram is adequate, so its states s_+ and s_- are adequate. Note that every non-alternating link diagram can be transformed into some alternating diagram and its mirror image by crossing changes which correspond to changes between positive and negative markers. Hence, two adequate states of a non-alternating diagram can be obtained by appropriate choice of markers corresponding to crossing changes transforming the non-alternating diagram to the alternating one. \square

The first link that has an adequate state other then s_+ and s_- is the knot $4_1(22)$ and it is illustrated in Fig. 6.

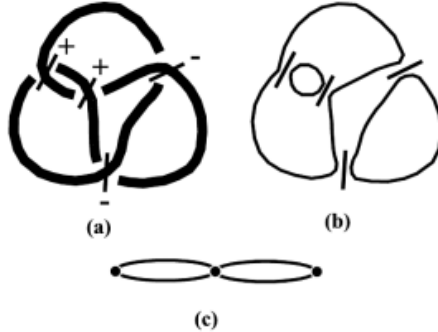


Figure 6: (a) Minimal diagram of the figure-eight knot with two +markers, and two -markers; (b) state circles; (c) the associated adequate graph G_s .

The minimal diagram of inadequate knot $20. - 3. - 20.2$ has as many as 11 adequate states. First two are obtained from the alternating diagram $20.3.20.2$ and its mirror image. The remaining nine adequate states can be obtained from other adequate diagrams, one corresponding to the minimal diagram $20. - 3. - 20. - 2$ and the other to the non-minimal diagram $-20.3.20. - 2$ which is reducible to 10-crossing non-alternating knot $10_{124}(5, 3, -2)$ (Fig. 7).

Theorem 18. Vertex connectivity of every adequate graph G_{s_+} or G_{s_-} corresponding to an alternating diagram D is greater then 1. Vertex connectivity of every

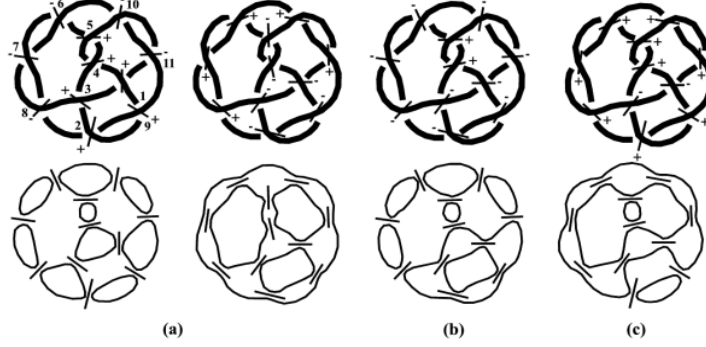


Figure 7: (a) Two adequate states of the inadequate knot diagram $20. - 3. - 20.2$ obtained from the alternating knot $20.3.20.2$; (b) adequate state of the same diagram corresponding to the minimal diagram $20. - 3. - 20. - 2$; (b) its adequate state corresponding to the non-minimal diagram $-20.3.20. - 2$, which is reducible to 10-crossing non-alternating knot $10_{124} (5, 3, -2)$.

adequate graph G_{s+} or G_{s-} corresponding to a non-alternating minimal diagram D is 1.

The same statement is not true for adequate graphs G_s obtained from other states. For example, the adequate graph G_s of the minimal non-alternating diagram of the knot $10_{155} = -3 : 2 : 2$ (Fig. 8) has the vertex connectivity 4.

Definition 8. The minimal number of adequate states taken over all diagrams of a link L is called the *adequacy number* of link L and denoted by $a(L)$.

Lemma 1. All minimal diagrams of the same alternating link have the same number of adequate states.

Since changing marker in one vertex is equivalent to the crossing change, we conclude that the number of adequate states is invariant of a link diagram independent from the signs of crossings. This means that the number of adequate states is the same for every alternating diagram and all non-alternating diagrams obtained from it by crossing changes. Moreover, this can be generalized to families of links, since adding a bigon to the chain of bigons does not change the adequacy of a diagram.

Lemma 2. The number of adequate states $a(L)$ is the invariant of a family of alternating links L and it is realized in any minimal diagram of the link family.

Theorem 19. The only links with $a(L) = 2$ are links of the family n ($n = 2, 3, 4, 5, \dots$), i.e., $2_1^2, 3_1, 4_1^2, 5_1, \dots$

The numbers of adequate states of two minimal diagrams of a non-alternating link can be different. The minimal diagram of the knot $3, 21, -2$ has 6 adequate states, and its another minimal diagram $.2. - 20. - 1 : . - 1$ has 8 adequate states, since the source link of the first diagram is $2, 21, 2$, and the source link of the other $.2.20$.

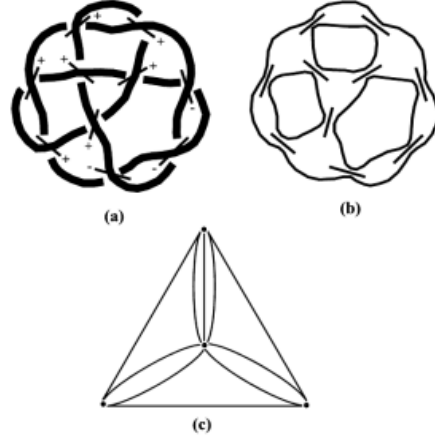


Figure 8: (a) Minimal diagram of the knot 10_{155} with markers; (b) state circles; (c) the associated graph G_s with the vertex connectivity 4.

Adequacy numbers of alternating link families obtained from source links with at most $n = 9$ crossings are given in the following table, where every family is represented by its source link.

$n = 2$	2 2					
$n = 4$	2 2 3					
$n = 5$	2 1 2 4					
$n = 6$	2 2 2 5	2 1 1 2 5	2, 2, 2 5			
$n = 7$	2 1 2 2 6	2 1 1 1 2 7	2, 2, 2+ 8	2 1, 2, 2 6	.2 7	
$n = 8$	2 2 2 2 8	2 1 2 1 2 8	2 2 1 1 2 8	2 1 1 1 1 2 9	2, 2, 2, 2 12	2 2, 2, 2 7
	2 1 1, 2, 2 9	2 1, 2 1, 2 8	2, 2, 2+ + 9	2 1, 2, 2+ 9	(2, 2) (2, 2) 8	.21 10
	.2 : 2 9	.2.2 8	.2 : 2 0 8	.2.2 0 8		
$n = 9$	2 2 1 2 2 9	2 2 2 1 2 10	2 1 2 1 1 2 10	2 2 1 1 1 2 11	2 1 1 1 1 1 2 12	2 1, 2 1, 2 1 12
	2 1 2, 2, 2 10	2 2 1, 2, 2 11	2 1 1 1, 2, 2 13	2 1, 2, 2, 2 9	2 2, 2 1, 2 11	2 1 1, 2 1, 2 10
	2 1, 2, 2+ + 10	2, 2, 2, 2+ 16	2 2, 2, 2+ 1 12	2 1 1, 2, 2+ 13	2 1, 2 1, 2+ 11	(2 1, 2) (2, 2) 11
	(2, 2) (2, 2) 10	(2, 2) 1 (2, 2) 12	.2.2 11	.2 1 1 13	.2 1 : 2 12	.2 1 : 2 0 12
	.2 1.2 0 11	.2.2 0.2 10	2 : 2 0 : 2 0 10	2 0 : 2 0 : 2 0 9	.2.2.2 10	2 : 2 : 2 9
	.2.2.2 0 9	2 : 2 : 2 0 9	.(2, 2) 14	8*2 12	8*2 0 13	

7. Adequacy polynomial as an invariant of alternating link families

Adequate state graphs corresponding to link diagrams can be used for creating a polynomial invariant of alternating link families.

Definition 9. A *cut-vertex* (or articulation vertex) of a connected graph is a vertex whose removal disconnects the graph [Char]. In general, a cut-vertex is a vertex of a graph whose removal increases the number of components [Har]. A graph with no cut-vertices is called a *biconnected graph* [Ski]. A *block* is a maximal biconnected subgraph of a given graph.

The following transformations will be applied to the adequate state graphs, till the graph cannot be reduced to a graph with lower number of vertices:

- (multiple edge reduction) replace every edge of the multiplicity greater than 2 by a single edge;^{||}
- (edge chain collapse) replace maximal part of every chain consisting from edges with vertices of degree 2 by a new edge connecting the beginning vertex of the first and ending vertex of the last edge;
- (block move) every block can be moved along the edges of the remaining part of the graph.

From every adequate state graph G we obtain the reduced adequate state graph \overline{G} .

Theorem 20. Block move preserves graph torsion and chromatic polynomial of a graph [PrPaSa].

Fig. 9 illustrates reduction of the graph with 16 vertices (Fig. 9a) to the graph with 13 vertices (Fig. 9b), or to its equivalent graph (Fig. 9c) obtained from it by block moves, which has the same torsion and chromatic polynomial as the graph (Fig. 9b).

Consider an arbitrary minimal diagram of an alternating link L . Let G_i denote the corresponding state graphs for all adequate states of a diagram D_L and \overline{G}_i reduced state graphs ($i = 1, 2, \dots, a(L)$), where $a(L)$ is the adequacy number of L (Def. 8).

Definition 10. The *adequacy polynomial* of any alternating diagram D_L is a polynomial in two variables determined by $A(x, y) = \sum_{i=1}^{a(L)} x^{\bar{t}_i} \overline{P}_i(y)$ where $\overline{P}_i(y) = P(\overline{G}_i)$ denotes the chromatic polynomial of a reduced state graph G_i and \bar{t}_i is the power of \mathbb{Z}_2 torsion of the first chromatic graph cohomology $H_{A_m}^{1,h}(G_i)$ over algebra of truncated polynomials $A_m = \mathbb{Z}[x]/x^2 = 0$ in the grading $h = (m-1)(v-2) + 1$ where v denotes number of vertices of the graph G_i .

Theorem 21. Adequacy polynomial is the same for all minimal diagrams of all alternating links belonging to the same family, which satisfy the condition $|a| + k_a \geq 3^{**}$.

The computation of adequacy polynomial is illustrated on the example of link $3\,1\,5\,4$. This link has 3 different minimal diagrams: $3\,1\,5\,4^{\dagger\dagger}$, $((1, (1, 3), 1, 1, 1, 1), 1, 1, 1, 1)$, and $((1, 1, (3, 1), 1, 1, 1), 1, 1, 1, 1)$ (Fig. 10). For the reduced adequate state graphs

^{||}Since chromatic polynomial of a graph and graph homology does not recognize mutiple edges, this step is not necessary for further computations [PrPaSa].

^{*}Please compare this additional condition with the definition of a family of link diagrams (Def. 5): according to the additional condition all chains of bigons must be of the length greater then 2.

^{††}This diagram can be also written as $((3, 1), 1, 1, 1, 1, 1), 1, 1, 1, 1)$.

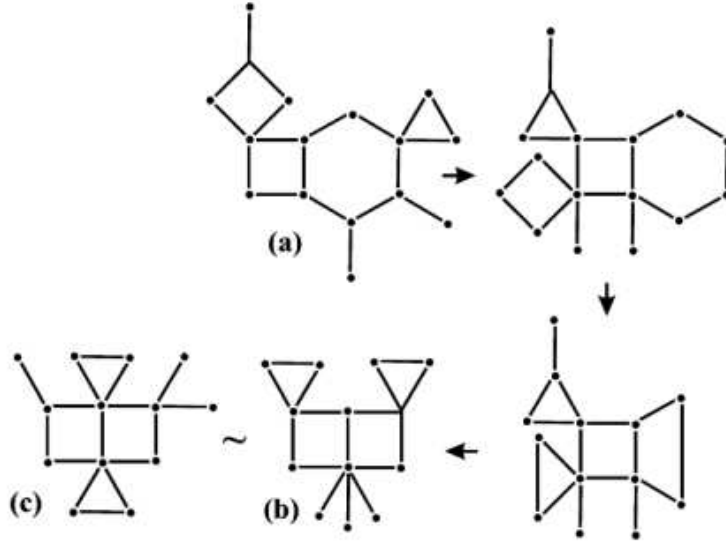


Figure 9: Reduction of the graph (a) to the graph (b), and graph (c) equivalent to (b).

\overline{G}_i ($i = 1, 2, \dots, 6$) corresponding to the first minimal diagram, the sequence $(1, 2, 2, 1, 1, 0)$ represents powers of \mathbb{Z}_2 -torsion \bar{t}_i for $m = 3$, and the following is the list of chromatic polynomials:

- 1) $6y - 15y^2 + 14y^3 - 6y^4 + y^5$,
- 2) $4y - 12y^2 + 13y^3 - 6y^4 + y^5$,
- 3) $-4y + 16y^2 - 25y^3 + 19y^4 - 7y^5 + y^6$,
- 4) $-18y + 81y^2 - 156y^3 + 168y^4 - 110y^5 + 44y^6 - 10y^7 + y^8$,
- 5) $-2y + 5y^2 - 4y^3 + y^4$,
- 6) $-9y + 27y^2 - 33y^3 + 21y^4 - 7y^5 + y^6$,

so the adequacy polynomial is

$$A(3154) = -9y - 14xy + 27y^2 + 71xy^2 + 4x^2y^2 - 33y^3 - 146xy^3 - 12x^2y^3 + 21y^4 + 163xy^4 + 13x^2y^4 - 7y^5 - 109xy^5 - 6x^2y^5 + y^6 + 44xy^6 + x^2y^6 - 10xy^7 + xy^8.$$

This polynomial is invariant of link family $p1qr$ ($p, q, r \geq 3$).

If we compute the adequacy polynomial from the second or third diagram, we obtain the same sequence $\bar{t}_1, \bar{t}_2, \dots, \bar{t}_6 = (1, 2, 2, 1, 1, 0)$ and the same list of chromatic polynomials, so the final result remains the same.

All minimal diagrams of the link family $p1qr$ ($p, q, r \geq 3$) have the same adequacy polynomial.

Conjecture 1. Adequacy polynomial distinguishes all alternating link families (up to mutation).

This conjecture is verified for all alternating links with at most $n = 12$ crossings. If the conjecture does not hold in general, one may consider various adequacy polynomials obtained by taking into consideration other gradings in first homology or the whole groups (possibly higher in homology) or changing algebra. Moreover, depending on the algebra one may consider torsions other than \mathbb{Z}_2 , if they exist.

Adequacy polynomial of any family of alternating links can be computed from a minimal diagram of the link L representing this family with all chains of bigons of the length 3. For subfamilies we use links with some parameters equal 2, and all the other equal 3. For the general Conway symbol $p\,1\,q\,r$ ($p, q, r \geq 2$), we need to distinguish the following cases:

1. $2\,1\,2\,2$ with $A(x, y) = y + 2xy - 2y^2 - 5xy^2 - 4x^2y^2 + y^3 + 6xy^3 + 8x^2y^3 - 4xy^4 - 5x^2y^4 + xy^5 + x^2y^5$;
2. $p\,1\,2\,2$ with $A(x, y) = -8y - 8xy + 25y^2 + 26xy^2 - 4x^2y^2 - 32y^3 - 33xy^3 + 8x^2y^3 + 21y^4 + 21xy^4 - 5x^2y^4 - 7y^5 - 7xy^5 + x^2y^5 + y^6 + xy^6$, $p \geq 3$;
3. $2\,1\,q\,2$ with $A(x, y) = -2xy - 4x^2y + 9xy^2 + 8x^2y^2 - 14xy^3 - 5x^2y^3 + 11xy^4 + x^2y^4 - 5xy^5 + xy^6$, $q \geq 3$;
4. $2\,1\,2\,r$ with $A(x, y) = 10xy - 8x^3y - 27xy^2 + 28x^3y^2 + 29xy^3 - 38x^3y^3 - 15xy^4 + 25x^3y^4 + 3xy^5 - 8x^3y^5 + x^3y^6$, $r \geq 3$;
5. $p\,1\,q\,2$ with $A(x, y) = -9y^2 - 2xy^2 - 4x^2y^2 + 27y^3 + 5xy^3 + 8x^2y^3 - 33y^4 - 4xy^4 - 5x^2y^4 + 21y^5 + xy^5 + x^2y^5 - 7y^6 + y^7$, $p, q \geq 3$;
6. $p\,1\,2\,r$ with $A(x, y) = -9y + 6xy + 16x^2y + 27y^2 - 17xy^2 - 60x^2y^2 - 33y^3 + 19xy^3 + 92x^2y^3 + 21y^4 - 10xy^4 - 75x^2y^4 - 7y^5 + 2xy^5 + 35x^2y^5 + y^6 - 9x^2y^6 + x^2y^7$, $p, r \geq 3$;
7. $2\,1\,q\,r$ with $A(x, y) = 8xy + 8x^2y - 20xy^2 - 32x^2y^2 + 20xy^3 + 54x^2y^3 - 10xy^4 - 50x^2y^4 + 2xy^5 + 27x^2y^5 - 8x^2y^6 + x^2y^7$, $q, r \geq 3$;
8. $p\,1\,q\,r$ with $A(x, y) = -9y - 14xy + 27y^2 + 71xy^2 + 4x^2y^2 - 33y^3 - 146xy^3 - 12x^2y^3 + 21y^4 + 163xy^4 + 13x^2y^4 - 7y^5 - 109xy^5 - 6x^2y^5 + y^6 + 44xy^6 + x^2y^6 - 10xy^7 + xy^8$, $p, q, r \geq 3$.

Without any changes, computation of adequacy polynomial can be extended to families of virtual links. Equivalents of Theorem 21 and Conjecture 1 hold for alternating virtual links.

The equivalent of Conjecture 1 is verified by computer calculations for all families of virtual knots derived from real knots with at most $n = 8$ crossings.

Definition of the adequacy polynomial (Def. 10) contains first chromatic graph homology in the specific grading coming from the interpretation of Hochschild homology as the chromatic graph homology of a polygon, i.e., $H_{A_m}^{1, (m-1)(v-2)+1}(G)$ where G is a graph and v denotes the number of its vertices and $A_m = \mathbb{Z}[x]/x^m$

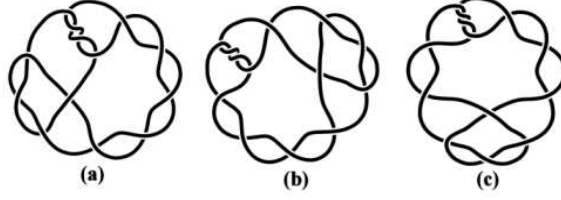


Figure 10: Minimal diagrams (a) 3 1 5 4; (b) $((1, (1, 3), 1, 1, 1, 1), 1, 1, 1, 1)$; (c) $((1, 1, (3, 1), 1, 1, 1), 1, 1, 1, 1)$.

for $m \geq 3$. The reason why we have excluded algebra A_2 is that it does not distinguish some generating links (e.g., 2 2 1 1 1 2 from 2 1 1, 2 1, 2). According to the computations for all generating links with $n \leq 12$ crossings, for $3 \leq m \leq 5$ adequacy polynomial distinguishes all families of alternating links with at most $n = 12$ crossings (up to mutation).

Notice that adequacy polynomials of the family 3 1 3 3 computed for $m = 2, 3, \dots, 8$ are the same, but this is not true on general: according to the computer calculations for $2 \leq m \leq 8$, the family $.p$ ($p > 2$) will have two different polynomials

$$2y - 10xy - 10x^2y - 4y^2 + 21xy^2 + 27x^2y^2 + 2y^3 - 14xy^3 - 31x^2y^3 + 3xy^4 + 20x^2y^4 - 7x^2y^5 + x^2y^6$$

for odd m , and

$$2y - 4xy - 10x^2y - 6x^4y - 4y^2 + 10xy^2 + 27x^2y^2 + 11x^4y^2 + 2y^3 - 8xy^3 - 31x^2y^3 - 6x^4y^3 + 2xy^4 + 20x^2y^4 + x^4y^4 - 7x^2y^5 + x^2y^6$$

for even m .

In the computation of adequacy polynomials we can also use second graph homology $H_{A_m}^{2, (m-1)(v-2)}(G)$, which for $m = 2, 4, 6$ distinguishes all alternating link families corresponding to links with at most $n = 12$ crossings.

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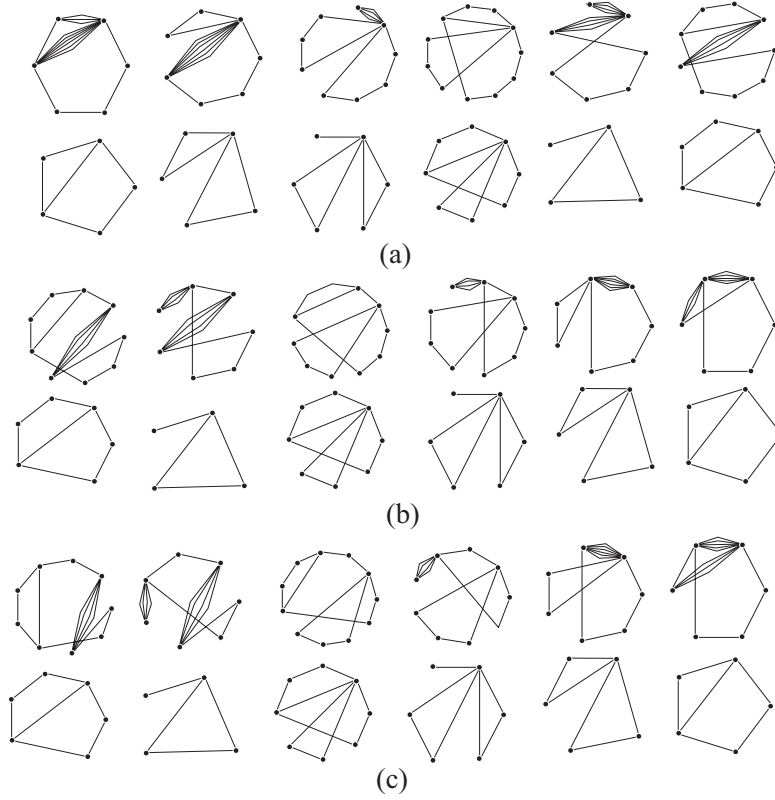


Figure 11: Adequate state graphs of the diagrams (a) $3\,1\,5\,4$, (b) $((1, (1, 3), 1, 1, 1, 1), 1, 1, 1, 1)$, (c) $((1, 1, (3, 1), 1, 1, 1), 1, 1, 1, 1)$ and their corresponding reduced graphs.

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